Last Time: Basics of Enumeration.

Correspondence Principle: If there is a one-to-one correspondence between sets $S \leftrightarrow T$, then $\#S = \#T$.

Addition Principle: If S and T are disjoint (i.e. $S \cap T = \emptyset$), then $\#(S \cup T) = (\#S) + (\#T)$.

Product Principle: $\#(S \times T) = (\#S) \cdot (\#T)$.

Prop: Every finite set S satisfies $\#\mathcal{P}(S) = 2^{\#S}$.

pf from last lecture (highlights): By induction.

base case: $\#S = 0$. ✓

Inductive step: $S \xleftrightarrow{f} [n]$.

$$\mathcal{P}(S) \leftrightarrow \mathcal{P}[n]$$

$$\mathcal{P}[n] = \underbrace{\{T \subseteq [n] : n \in T\}}_{A} \cup \underbrace{\{T \subseteq [n] : n \notin T\}}_{B}$$

disjoint union!

$\updownarrow$

$\{T' \subseteq [n-1] : \text{true}\} = \mathcal{P}[n-1]$

Correspondence: $T' = T \setminus \{n\}$

$=$

$\{T \subseteq [n] : T \subseteq [n-1]\} = \mathcal{P}[n-1]$

$(T \in A,\ U \in B)$

$\to n \in T,\ n \notin U$

$\to T \neq U$

$\to A \cap B = \emptyset$

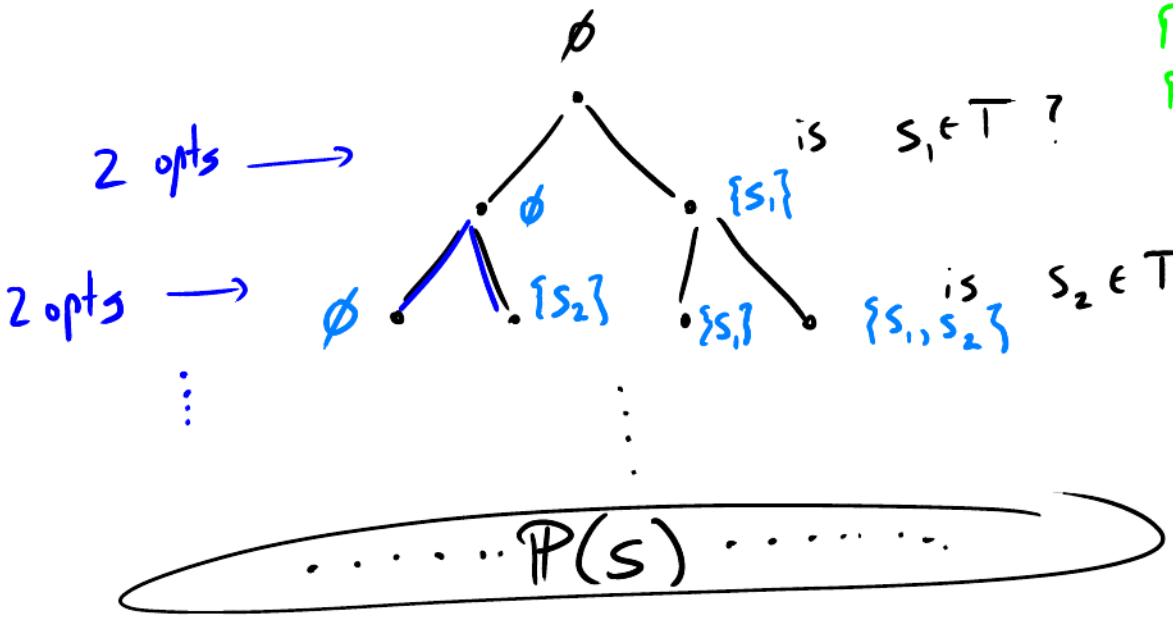
$\therefore \#\mathcal{P}[n] = \#A + \#B = \#\mathcal{P}[n-1] + \#\mathcal{P}[n-1]$

$= 2^{n-1} + 2^{n-1} = 2^n$ ∎

Second Sketch: If S is finite, list elements of S in roster notation: $S = \{s_1, s_2, \dots, s_n\}$.

Each subset $T \subseteq S$ can be built by making some choices (illustrate w/ decision tree):

For each $s \in S$, answer "include s in T?"



$\rightsquigarrow$ $\therefore \#\mathbb{P}(S) = 2^n$...

(NB: This argument is not rigorously made!).

Prop: Every finite S has $\#\mathbb{P}(S) = 2^{\#S}$.

Pf: Let S be a set of size $n \in \mathbb{N}$. Note this implies S is in one-to-one correspondence w/ $[n]$. This yields a one-to-one correspondence of $\mathbb{P}(S)$ with $\mathbb{P}[n]$, so $\#\mathbb{P}(S) = \#\mathbb{P}[n]$.

Now proceed by induction on n.

Base Case: When $n = 0$, $[n] = \emptyset$, and we see

$$\#\mathbb{P}([n]) = \#\mathbb{P}(\emptyset) = \#\{\emptyset\} = 1 = 2^0 = 2^n.$$

Inductive Step: Suppose $\#\,\mathbb{P}[n-1] = 2^{n-1}$. Note every element of $\mathbb{P}[n]$ can be constructed by the following sequence of choices:

① Choose $T \subseteq [n-1]$. (2^{n-1} options)

② Choose to include or exclude n. (2 options).

Note that these choices uniquely determine every element of $\mathbb{P}[n]$. Indeed, Given $T' \in \mathbb{P}[n]$, Let $T := T' \cap [n-1]$, and note either $n \in T$ or $n \notin T$. This determines the choice sequence for T'.

To see uniqueness, If $U', T' \in \mathbb{P}[n]$ and $U' = T'$, The choices $U = U' \cap [n-1]$, $T = T' \cap [n-1]$ and (is $n \in U'$?), (is $n \in T'$), must agree!

Hence, by the product principle we have

$$\#\,\mathbb{P}(S) = \#\,\mathbb{P}[n] = (\#\,\mathbb{P}[n-1]) \cdot 2 = 2^{n-1} \cdot 2 = 2^n.$$

Hence the claim holds by weak induction. ☒

Answer to Chat: $P(n-1) \to P((n-1)+1)$

$K \to K+1$ ←

~~$K \to K-1$~~ doesn't work.

What else can we count?

Goal: Enumerate $\{S \subseteq [n] : \#S = k\}$ for all $n \geq k \geq 0$ ← hard w/o a clever rewording...

Def^n: Let S be a set. A <u>k-sequence</u> in S is a <u>k-tuple</u> $(s_1, s_2, s_3, \dots, s_k)$ where (k elts.)

$s_i \in S$ for all $1 \le i \le k$.

Remark: For k-sequences, order matters ← counting easy
For k-sets, order doesn't matter ← counting hard

Ex: $S = [5]$. $(1,2,3)$ and $(1,3,2)$ are <u>different</u> 3-sequences, whereas $\{1,2,3\}$ and $\{1,3,2\}$ are the same S.

Point: Can count k-sequences using the product principle. Can't do that (directly) for k-sets...

Ex: $(1,2,2)$ is a 3-sequence in $[7]$, whereas $\{1,2,2\} = \{1,2\}$ is a 2-set, <u>not</u> a 3-set.

Prop: Given set S of size n, there are n^k k-sequences in S.

Sketch: We can construct every k-sequence in S uniquely via the following choices:

① choose $s_1 \in S$ ($\#S = n$ options)
② choose $s_2 \in S$ (n options)
⋮
ⓚ choose $s_k \in S$ (n options)

Result is $(s_1, s_2, \dots, s_k)$. Hence n^k total possibilities ☒

Exercise: Prove it ↑ by induction (using our sketch as guide).

Defn: An ordered k-subset of S is a k-sequence $(s_1, s_2, \cdots, s_k)$ in S such that

$s_i = s_j$ implies $i = j$.

NB: The above condition "$s_i = s_j$ implies $i = j$" means there are no repeats in the k-sequence.

In the special case $k = \#S$, we say an ordered k-subset is called a permutation of S.

Prop: A finite set of size n has $n!$ permutations.

pf: Replacing S by $[n]$, we induct on n.

Base Case: When $n=0$, we have exactly one 0-sequence in $[0] = \emptyset$: $()$. This is the unique permutation of $\emptyset$, so $1 = 0!$ agrees with our count.

Inductive Step: Suppose the result holds for $[n-1]$.

I.e. we have $(n-1)!$ permutations of $[n-1]$.

Every permutation of $[n]$ can be constructed uniquely via the following two choices.

① Choose a permutation $\sigma = (k_1, k_2, \cdots, k_{n-1})$ of $[n-1]$. ($(n-1)!$ options)

② Choose a position i among $1, 2, \cdots, n$ to place n. (n options)

The resulting permutation is obtained by placing n in

the i'th position of σ, shifting all subsequent entries of σ up. (NB: no need to shift if $i=n$).

Hence, by the product principle, the number of permutations of $[n]$ is precisely $(n-1)! \cdot n = n!$

Hence the original statement holds by induction. ☒

Prop: There are $\frac{n!}{(n-k)!}$ ordered k-subsets of $[n]$.

NB: We can deal w/ the problem that there is a division in this expression by simply clearing the denominator...

Pf: Let $n \in \mathbb{N}$ and $0 \leq k \leq n$. Let S denote the set of ordered k-subsets of $[n]$. We note that every permutation of $[n]$ can be built uniquely via the following choices:

① Choose an ordered k-subset of $[n]$, $(a_1, a_2, \cdots, a_k)$. (#S options).

② Choose a permutation $(b_1, b_2, \cdots, b_{n-k})$ of $[n] \setminus \{a_1, a_2, \cdots, a_k\}$ ($(n-k)!$ options)

The resulting permutation of $[n]$ is

$$(a_1, a_2, \cdots, a_k, b_1, b_2, \cdots, b_{n-k})$$

(Left to you: this does construct all permutations uniquely...).

Hence we have two counts for the number of permutations of $[n]$; as they must agree, we see

$$n! = \#\{\text{permutations of } [n]\} = \#S \cdot (n-k)!.$$

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